Pergamon

Int. J. Solids Structures Vol. 34, No. 4, pp. 435–450, 1997
Published by Elsevier Science Ltd
Printed in Great Britain
0020–7683/97 $17.00 + .00

PII: S0020–7683(96)00019–4

ON SHEAR BAND NUCLEATION AND THE FINITE PROPAGATION SPEED OF THERMAL DISTURBANCES

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(*Received* 30 *May* 1995; *in revised form* 8 *January* 1996)

Abstract—This paper examines the onset of shear flow localization in thermal viscoplastic materials at high rates of loading while accounting for the finite speed of thermal wave propagation. It is well known that the Fourier's classical law of heat conduction implies that thermal disturbances travel with infinite speed. We introduce a modification to the classical Fourier law which renders a finite speed for thermal wave propagation. We then utilize the energy viewpoint of dynamic localization introduced in Shawki (Shawki, T. G. (1994). An energy criterion for the onset of shear localization in thermal viscoplastic materials, part I: Necessary and sufficient initiation conditions. *ASME J. Appl. Mech.* **61,** 530–537) and (Shawki, T. G. (1994). An energy criterion for the onset of shear localization in thermal viscoplastic materials, part II: Applications and implications. *ASME J. Appl. Mech.* **61,** 538–547) as well as a linear stability analysis to determine the necessary conditions for the onset of localization. An exact linear solution is obtained for adiabatic deformations of thermal viscoplastic materials with no strain dependence. A matched asymptotic expansion is obtained for the general case involving non-adiabatic deformations of the foregoing class of materials. The foregoing solutions illustrate the effect of finite speed of thermal wave propagation on localization initiation. Published by Elsevier Science Ltd.

1. INTRODUCTION

It is well known that the commonly used heat conduction equation based on the classical Fourier's law is a parabolic partial differential equation which exhibits the undesirable feature of predicting that thermal disturbances propagate with infinite speeds. Hence, a sudden localized heat generation due to plastic deformation is instantaneously sensed throughout the deforming body. The infinite speed of propagation of thermal disturbances is physically unrealistic and may lead to erroneous conclusions when the time needed for the decay of thermal transients is of the same order as the times at which certain physical quantities are measured in a thermomechanical process.

To overcome this shortcoming, several modifications to the classical heat equation have been proposed. Such modifications may be broadly classified into two main categories: (1) modification of the Fourier law and the concept of a thermal relaxation time and (2) introduction of the temperature rate as an internal state variable in the constitutive equations while still using the classical Fourier law. An extensive review of these two theories is provided by Chandrasekharaiah (1986).

Category (1) has received wider acceptance in view of its simplicity. The commonly used form of this theory produces a *hyperbolic* heat conduction equation which results in finite wave speeds for thermal disturbances. Achenbach (1968) considered the problem of thermal and mechanical waves propagating in a one-dimensional semi-infinite medium due to the sudden application of mechanical and thermal disturbances at the free surface. Kim and Hector (1991) considered the problem of hyperbolic heat conduction in materials subjected to laser radiation where the times of observation are of the order of nanoseconds. The usefulness of the modified heat conduction equation in materials with an inhomogeneous inner structure is discussed by Kaminski (1990). Experimental methods for the determination of thermal wave speeds are discussed by Gembaroric and Majernik (1987). Mason and Rosakis (1992) utilized the hyperbolic heat equation to determine the temperature field in the neighbourhood of the tip of a dynamically propagating crack.

This paper examines the implications of the finite propagation speed of thermal disturbances as regards the onset of shear localization in dynamic deformations of thermal viscoplastic materials. This study is conducted in the context of a one-dimensional simple shearing motion. This study is motivated by the fact that observations of adiabatic shear bands are reported at ultra-high loading rates. For example, typical strain rates associated with Kolsky bar experiments are in the range $[10^2\text{–}10^4]$ sec^{-1}. Furthermore, the strain rates achieved in plate impact experiments belong to the range $[10^5\text{–}10^7]$ sec^{-1}. Higher strain rates are reported in ballistic impact problems. Subject to such high strain rates, deformations in ductile materials are highly dissipative and involve the release of large amounts of energy within very short time intervals. Moreover, the formation of adiabatic shear bands in such deformations leads to greater rates of thermal energy generation within the localized zones. The high rates of thermal energy generation produce temperature increases of two to three times in few microseconds. As pointed out by Chandrasekharaiah (1986), for such high rates of deformation, the classical heat conduction equation may not provide an adequate account of thermal energy transfer.

The primary mechanism underlying shear band formation at high loading rates is believed to be a thermo-mechanical mechanism in which most of the generated plastic work is converted to heat. The generation of heat is highest at sites of maximum plastic strain rate. The spatially-inhomogeneous generation of heat coupled with material's thermal softening leads to the evolution of a spatially inhomogeneous plastic strain rate field. Since the generation of heat combined with the lack of time for heat conduction are the main building blocks for this mechanism; it seems appropriate to examine the issues related to the time scale of heat transfer. In this paper, we confine our attention to the onset conditions for shear band formation while accounting for the finite wave speed for the propagation of thermal energy. Hence, we utilize the linear stability theory developed in Shawki (1994a and 1994b) and use the energy-based framework for the characterization of localization. We derive the necessary conditions for localization and compare the results to earlier results obtained by using the classical Fourier's law of heat conduction.

It is useful to note that the issue as to whether or not thermal energy propagate with finite speed is not completely resolved. The issue appears to be whether or not the modified energy equation can be justified from fundamental quantum mechanics considerations and whether or not thermal energy propagates as a wave. In this work, we are not addressing this particular issue. Instead, we are only interested in the implications of such effect on localization initiation conditions.

2. THE MODIFIED FOURIER LAW OF HEAT CONDUCTION

We follow the model developed by Gurtin and Pipkin (1968) in which the classical heat conduction equation is replaced by the modified equation

$$\left(1+\hat{t}_r\frac{\partial}{\partial\hat{t}}\right)\hat{q} = -\hat{\kappa}\frac{\partial\hat{\theta}}{\partial\hat{x}}, \tag{1}$$

where $\hat{t}_r$ is referred to as the relaxation constant, $\hat{q}$ is the heat flux, $\hat{\theta}$ is the absolute temperature and $\hat{\kappa}$ is the material thermal conductivity. Equation (1) implies that the heat flux accumulates over a finite time interval in the presence of a temperature gradient. As the temperature gradient vanishes the heat flux decays exponentially. Gurtin and Pipkin (1968) estimated the value of the relaxation constant $\hat{t}_r$ to be

$$\hat{t}_r = \frac{3\hat{\kappa}}{\hat{c}\hat{S}^2}, \tag{2}$$

where $\hat{c}$ is the specific heat of the solid per unit volume and $\hat{S}$ is the sound (phonon) velocity. Next, we derive the energy balance field equation using the first law of thermodynamics along with the modified Fourier's law given by eqn (1).

2.1. *The modified energy equation*

In view of our current objective related to adiabatic shear band formation, we make the following assumptions: (1) no internal heat sources in the deforming body, (2) no body forces, (3) material properties are homogeneous and temperature-independent, (4) elastic effects are ignored, and (5) the specific internal energy e is given by $c\theta$. Application of the first law of thermodynamics provides the energy balance equation

$$\hat{\rho}\frac{\mathrm{d}\hat{e}}{\mathrm{d}\hat{t}} = \hat{T}:\hat{D}-\nabla\cdot\hat{q}, \tag{3}$$

where $\hat{e}$ is the specific internal energy, $\hat{\rho}$ is the material density, $\hat{T}$ is the Cauchy stress tensor, $\hat{D}$ is the rate of deformation tensor and $\hat{q}$ is the heat flux vector. The operator $\mathrm{d}/\mathrm{d}t$ denotes the material time derivative. We now specialize eqn (3) to the case corresponding to one-dimensional simple shearing motion which is schematically illustrated in Fig. 1. The only non zero velocity component is $\hat{v}_y \equiv \hat{v}$ while all field variables are assumed to only depend on the space direction $\hat{x}$. The normal stresses are shown by Shawki and Clifton (1989) to have a weak effect as regards shear band formation. Therefore, the relevant energy balance for the considered simple shearing motion is given by

$$\hat{\rho}\hat{c}\frac{\partial\hat{\theta}}{\partial\hat{t}} = \beta\hat{\sigma}\frac{\partial\hat{v}}{\partial\hat{x}} - \frac{\partial\hat{q}}{\partial\hat{x}}, \tag{4}$$

where $\hat{\sigma}$ is the shear stress ($\hat{\sigma}_{\hat{x}\hat{y}}$). Upon substitution of eqn (1) into the energy balance equation (4), one obtains

$$\frac{\partial\hat{\theta}}{\partial\hat{t}} - \hat{r}_1\hat{\sigma}\frac{\partial\hat{v}}{\partial\hat{x}} - \hat{r}_0\frac{\partial^2\hat{\theta}}{\partial\hat{x}^2} = \hat{t}_r\left[\hat{r}_1\frac{\partial}{\partial\hat{t}}\left(\hat{\sigma}\frac{\partial\hat{v}}{\partial\hat{x}}\right) - \frac{\partial^2\hat{\theta}}{\partial\hat{t}^2}\right], \tag{5}$$

where $\hat{r}_1 = \beta/\hat{\rho}\hat{c}$ and $\hat{r}_0 = \hat{\kappa}/\hat{\rho}\hat{c}$. It is evident that as the relaxation constant $\hat{t}_r$ vanishes, eqn (5) reduces to the classical energy balance field equation without heat sources. Since the modified form (5) involves second temporal derivatives, it follows that a larger set of initial conditions must be specified. Furthermore, we note that in the absence of plastic dissipation; i.e., for $\hat{r}_1 = 0$, eqn (5) reduces to the following hyperbolic–parabolic equation:

$$\hat{t}_r\frac{\partial^2\hat{\theta}}{\partial\hat{t}^2} + \frac{\partial\hat{\theta}}{\partial\hat{t}} = \hat{r}_0\frac{\partial^2\hat{\theta}}{\partial\hat{x}^2}. \tag{6}$$

Examination of eqn (5) or eqn (6) implies that the ratio $\hat{r}_0/\hat{t}_r \equiv \hat{c}_{\hat{\theta}}^2$ plays the role corresponding to the square of the thermal wave speed. This renders the rate of heat propagation finite as long as the relaxation constant $\hat{t}_r$ remains strictly positive.

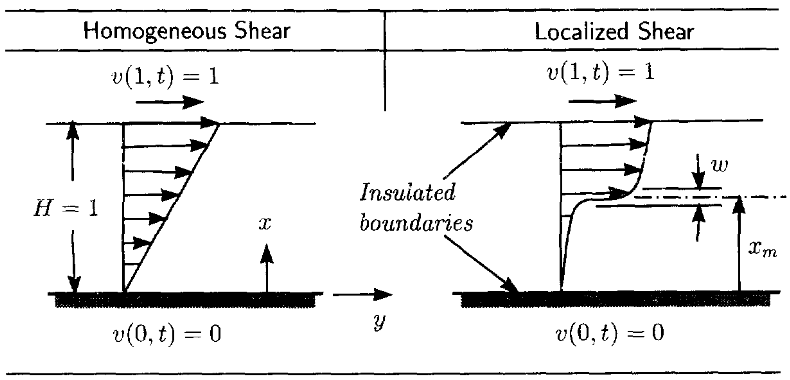

Fig. 1. A schematic of one-dimensional simple shearing motion.

3. PROBLEM FORMULATION

This section presents a brief account of the governing equations for the simple shearing motion of an infinite plate of thickness H subject to constant prescribed velocities as illustrated in Fig. 1. It is assumed that the problem is one-dimensional in the sense that all the quantities are only functions of $\hat{x}$ and $\hat{t}$. Further, we consider a homogeneous material and ignore elastic effects. A constant velocity $\hat{V}_0$ is prescribed at the upper plate boundary while the lower plate boundary is fixed. The system of equations is given by

$$\hat{\rho}\frac{\partial \hat{v}}{\partial \hat{t}} = \frac{\partial \hat{\sigma}}{\partial \hat{x}}, \tag{7}$$

$$\frac{\partial \hat{v}}{\partial \hat{x}} = \hat{\dot{\gamma}}^p, \tag{8}$$

$$\frac{\partial \hat{\theta}}{\partial \hat{t}} - \hat{r}_1\hat{\sigma}\frac{\partial \hat{v}}{\partial \hat{x}} - \hat{r}_0\frac{\partial^2 \hat{\theta}}{\partial \hat{x}^2} = \hat{t}_r\left[\hat{r}_1\frac{\partial}{\partial \hat{t}}\left(\hat{\sigma}\frac{\partial \hat{v}}{\partial \hat{x}}\right) - \frac{\partial^2 \hat{\theta}}{\partial \hat{t}^2}\right], \tag{9}$$

$$\hat{\dot{\gamma}}^p = \hat{\phi}(\hat{\sigma}, \gamma^p, \hat{\theta}) \quad \text{or} \quad \hat{\sigma} = \hat{\psi}(\hat{\dot{\gamma}}^p, \hat{\theta}, \gamma^p) \tag{10}$$

where eqn (7) is the balance of linear momentum, eqn (8) the kinematic compatibility equation, eqn (9) is the modified energy equation and eqn (10) provides alternative forms of the description of the material's thermal viscoplastic response.

The superposed "hat" in the governing equations denotes "dimensional" quantities. Here, $\hat{v}$ is the particle velocity in the y direction, $\hat{\sigma}$ the shear stress, $\hat{\theta}$ the absolute temperature, and $\hat{\gamma}^p$ the plastic strain. Furthermore, $\hat{\rho}$ the mass density, $\hat{\kappa}$ the constant heat conductivity coefficient, and $\hat{c}$ is a constant specific heat. A superposed "dot" denotes partial differentiation with respect to the time t. Upon appropriate normalization, the equations retain the same form while the hats are dropped. The dimensionless variables are defined by

$$t = \hat{t}\hat{V}_0/H, x = \hat{x}/H, \sigma = \hat{\sigma}/\hat{\sigma}_0, \dot{\gamma}^p = \hat{\dot{\gamma}}^p H/\hat{V}_0, \theta = \hat{\theta}/\hat{\theta}_0, t_r = \hat{t}_r\hat{V}_0/H, \tag{11}$$

$$\kappa = \hat{\kappa}\hat{\theta}_0/\hat{\sigma}_0 H\hat{V}_0, \rho = \hat{\rho}\hat{V}_0^2/\hat{\sigma}_0, c = \hat{c}\hat{\theta}_0/\hat{V}_0^2, \mu = \hat{\mu}/\hat{\sigma}_0, r_0 = \hat{\kappa}/\hat{\rho}\hat{c}H\hat{V}_0. \tag{12}$$

In eqns (11) and (12), $(\hat{V}, \hat{\sigma}_0, \hat{\theta}_0)$ denote the constant applied velocity, the reference flow stress and the reference temperature, respectively. The dimensionless form of the equations will be used throughout this document. The boundary conditions are given by

$$v(0,t) = 0, \quad v(1,t) = 1, \quad 0 \leqslant t \leqslant \infty \tag{13}$$

$$\theta_{,x}(0,t) = \theta_{,x}(1,t) = 0, \quad 0 \leqslant t \leqslant \infty. \tag{14}$$

In eqn (14), the notation $u_{,x}$ denotes partial differentiation with respect to the subscripted variable; that is $u_{,x} \equiv \partial u/\partial x$. The boundary conditions given by eqn (14) reflect thermally-insulated boundaries; i.e., adiabatic boundary conditions.

4. ENERGY-BASED LINEAR STABILITY ANALYSIS

The linear stability theory has been a powerful tool for studies of shear flow localization. Useful information regarding the necessary conditions for the onset of shear localization can be extracted from a linear stability approach to the considered boundary-value problem. Shawki (1994a and 1994b) has presented an energy-based theory of localization based on the linear stability approach. The procedure involves the linearization of the governing equations about a spatially-homogeneous solution and the subsequent examination of the temporal behavior of the $L2$-norm of the linear solution. Shawki (1994a) points out that the onset of shear localization is tied to positive rates of growth of the kinetic energy of the absolute perturbations. Here, we utilize this approach towards the boundary-value problem given by eqns (7–10) along with the boundary conditions (13, 14).

4.1. *The homogeneous solution*

We seek the derivation of a spatially-homogeneous solution of the considered boundary-value problem for an appropriate set of homogeneous initial conditions. It is straightforward to verify that the homogeneous solution for the particle velocity, the plastic strain rate and the plastic strain is given by

$$\bar{v}(x,t) = x, \tag{15}$$

$$\dot{\bar{\gamma}}^p(t) = 1, \tag{16}$$

$$\bar{\gamma}^p(t) = \gamma_0 + t. \tag{17}$$

The homogeneous temperature is the solution to the following non-autonomous ordinary differential equation:

$$\frac{d\bar{\theta}(t)}{dt} = C_1 e^{-t/t_r} + r_1\psi(1, \gamma_0 + t, \bar{\theta}(t)), \tag{18}$$

where

$$C_1 \equiv \bar{\theta}(0) - r_1\bar{\sigma}(0).$$

A superposed "bar" denotes a homogeneous solution. Note that taking account of the finite speed of thermal wave motion requires the prescription of an additional initial condition as opposed to the case associated with the classical Fourier law of heat conduction. Here, we choose $\bar{\theta}(0) = r_1\bar{\sigma}(0)$ so that $C_1 = 0$. This particular choice renders a homogeneous solution which is identical to that associated with the same initial boundary-value problem but with the classical Fourier law. We make this selection in order to obtain a consistent comparison of localization initiation in the two cases involving infinite or finite speed of thermal wave propagation.

5. LOCALIZATION CRITERION

We seek the determination of necessary conditions for the onset of shear localization while accounting for the finite speed of thermal wave propagation. Hence, we utilize the linear stability theory within the energy-based framework of localization presented by Shawki (1994a and 1994b).

5.1. *Linear perturbation analysis*

We seek a solution of the governing system of eqns (7–10) which satisfies the boundary conditions (13, 14) and has the following form:

$$v(x,t) = x + \tilde{v}(x,t), \tag{19}$$

$$\dot{\gamma}^p(x,t) = 1 + \dot{\tilde{\gamma}}^p(x,t), \tag{20}$$

$$\gamma^p(x,t) = (\gamma_0 + t) + \tilde{\gamma}^p(x,t), \tag{21}$$

$$\theta(x,t) = \bar{\theta}(t) + \tilde{\theta}(x,t), \tag{22}$$

$$\sigma(x,t) = \psi(1, \gamma_0 + t, \bar{\theta}) + \tilde{\sigma}(x,t), \tag{23}$$

where the over tildes represent the perturbations superposed on the homogeneous solution. The function $\bar{\theta}(t)$ in eqn (22) is the solution to the differential eqn (18) with the special choice of $C_1 = 0$. Such a linear decomposition of the solution represents a reasonable approximation to the nonlinear solution as long as the perturbations remain much smaller than the corresponding homogeneous quantities; i.e.,

$$|\tilde{v}(x,t)| \ll 1, \quad \left|\frac{\tilde{\theta}(x,t)}{\bar{\theta}(t)}\right| \ll 1, \quad \left|\frac{\tilde{\sigma}(x,t)}{\bar{\sigma}(t)}\right| \ll 1, \quad \left|\frac{\tilde{\gamma}^p(x,t)}{\gamma_0 + (t)}\right| \ll 1,$$

$$|\tilde{\dot{\gamma}}^p(x,t)| \ll 1, \quad \forall\, 0 \leqslant t < T \tag{24}$$

where T denotes the time at which the first violation of any of the inequalities in (24) occurs. In other words, T represents the time at which the linear solution is no longer valid.

We further note that the perturbations must have the form

$$\begin{aligned} \tilde{v}(x,t) &= V_n(t)\sin(\xi_n x), \quad \tilde{\gamma}(x,t) = \Gamma_n(t)\cos(\xi_n x), \\ \tilde{\sigma}(x,t) &= \Sigma_n(t)\cos(\xi_n x), \quad \tilde{\theta}(x,t) = \Theta_n(t)\cos(\xi_n x), \end{aligned} \tag{25}$$

where $\xi_n = n\pi, n = 1, 2, 3, \ldots$ is the wave number. The form (25) is selected for consistency with the prescribed boundary conditions. Substitution of eqns (19–23) and (25) into the governing system of nonlinear partial differential eqns (7–10) while retaining only first order terms of the perturbations renders a system of linear ordinary differential equations for the perturbation amplitudes. This system is given by

$$\rho \dot{V}_n(t) = -\xi_n \Sigma_n(t), \tag{26}$$

$$\xi_n V_n(t) = \dot{\Gamma}(t), \tag{27}$$

$$t_r \ddot{\Theta}_n(t) + \dot{\Theta}_n(t) + \alpha_n \Theta_n(t) = r_1 \left[\Sigma_n(t) + \bar{\sigma}(t)\dot{\Gamma}_n(t) + t_r \left(\dot{\Sigma}_n(t) + \dot{\overline{\bar{\sigma}(t)\dot{\Gamma}_n(t)}} \right) \right], \tag{28}$$

$$\Sigma_n(t) = \bar{S}_1(t)\dot{\Gamma}_n(t) + \bar{S}_2(t)\Theta_n(t) + \bar{S}_3(t)\Gamma_n(t), \tag{29}$$

where $\alpha_n \equiv r_0 \xi_n^2$ is the so-called local adiabaticity parameter and the sensitivities S_k, $k = 1, 2, 3$ are defined by

$$S_1 = \frac{\partial \psi}{\partial \dot{\gamma}^p}, \quad S_2 = \frac{\partial \psi}{\partial \theta}, \quad S_3 = \frac{\partial \psi}{\partial \gamma^p}.$$

The above measures reflect the material's strain rate sensitivity, thermal sensitivity and strain sensitivity, respectively. We are concerned with materials for which $S_1 > 0, S_2 \leqslant 0$ and $S_3 \geqslant 0$.

Here, we consider the case of a quasi-static deformation of a strain-independent material. Such a case may be mathematically realized by taking the limit $\rho \to 0$ and letting $S_3 = 0$. The appropriateness of the quasi-static analysis as far as deriving necessary conditions for the *onset* of localization is thoroughly discussed by Shawki (1994b). The consideration of a strain independent material response is motivated by the fact that strain hardening presents a stabilizing effect as far as localization evolution is concerned. Furthermore, the absence of strain dependence allows for the derivation of exact localization criterion through the energy-based linear stability analysis.

Examination of eqn (26) indicates that the stress amplitude $\Sigma_n(t)$ must vanish for a quasi-static deformation. Utilizing this result while setting $\bar{S}_3(t) = 0$ in eqn (29) gives

$$\dot{\Gamma}_n(t) = \bar{F}_2(t)\Theta_n(t), \tag{30}$$

where $F_2 = -S_2/S_1 = \partial\phi/\partial\sigma$ is an alternative measure of thermal sensitivity (note that $F_2 > 0$ for a thermally softening response). Substitution of the relationship (30) is eqn (28) and further rearrangement yields

$$t_r\ddot{\Theta}_n(t)+[1+t_r\bar{G}(t)]\dot{\Theta}_n(t)+[\alpha_n+\bar{G}(t)+t_r\dot{\bar{G}}(t)]\Theta_n(t) = 0, \tag{31}$$

where

$$G \equiv \frac{r_1\sigma S_2}{S_1} = -r_1\sigma F_2. \tag{32}$$

The quotient G represents the ratio of the slope, $C_p = S_3+r_1\sigma S_2$, of the adiabatic stress–strain curve at constant strain rate to the material's strain rate sensitivity. This ratio often plays a significant role in various analyses of dynamic localization in viscoplastic materials. Here, we reiterate that the barred quantities refer to evaluations at the homogeneous, time dependent solution.

At this point, we follow the framework developed by Shawki (1994a and 1994b) which associates the onset of shear localization with the positive rate of growth of the kinetic energy of the absolute perturbations. This criterion can be expressed in the useful form (see Shawki (1994b) for a detailed derivation)

$$\dot{\Gamma}_n(t)\ddot{\Gamma}_n(t) > 0. \tag{33}$$

Taking advantage of the relationship given by eqn (30) reduces the criterion (33) to

$$\frac{\dot{\bar{F}}_2(t)}{\bar{F}_2(t)}+\frac{\dot{\Theta}_n(t)}{\Theta_n(t)} > 0. \tag{34}$$

Equation (34) is the operative localization criterion applicable to the case considered in the present work. Knowledge of the quotient $\dot{\Theta}_n(t)/\Theta_n(t)$ in terms of quantities evaluated at the homogeneous solution and the relaxation time is our main objective.

5.2. *Exact solution for adiabatic deformations*

An exact solution can be obtained for the case of an adiabatic deformation for which $r_0 = 0$. We set $\alpha_n = 0$ in eqn (31) and divide both sides by the temperature perturbation amplitude to obtain

$$t_r\frac{\ddot{\Theta}_n(t)}{\Theta_n(t)}+[1+t_r\bar{G}(t)]\frac{\dot{\Theta}_n(t)}{\Theta_n(t)}+[\bar{G}(t)+t_r\dot{\bar{G}}(t)] = 0. \tag{35}$$

We now introduce the variable $\dot{\Theta}_n(t)/\Theta_n(t) \equiv Q_n(t)$ which transforms eqn (35) into

$$t_r(\dot{Q}_n(t)+Q_n^2(t))+[1+t_r\bar{G}(t)]Q_n(t)+[\bar{G}(t)+t_r\dot{\bar{G}}(t)] = 0. \tag{36}$$

The above equation is a first order, nonlinear ordinary differential equation which belongs to the class of equations known as Riccatti equations. It is useful to note that for the case of a classical Fourier's law of heat conduction, $t_r = 0$, and eqn (36) renders the exact solution $Q_n(t) = -\bar{G}(t)$. In this case, the operative localization criterion is

$$\frac{\dot{\bar{F}}_2(t)}{\bar{F}_2(t)} + r_1 \sigma \bar{F}_2(t) > 0, \tag{37}$$

which is identical to the condition derived by Shawki (1994b). For the case when the relaxation time is strictly positive, it can be easily verified that the function "$-\bar{G}(t)$" is a particular solution which satisfies eqn (36). This suggests the transformation

$$Q_n(t) = -\bar{G}(t) + \frac{1}{f_n(t)}, \tag{38}$$

which provides a linear ordinary differential equation, with variable coefficients, for the unknown function $f_n(t)$. This equation is given by

$$\dot{f}_n(t) + \left(\bar{G}(t) - \frac{1}{t_r}\right) f_n(t) = 1. \tag{39}$$

The exact solution of eqn (39) is obtained in terms of quadratures and substituted in expression (38) to obtain the desired solution for the quotient $Q_n(t)$ which is given by

$$Q_n(t) = -\bar{G}(t) + \frac{\exp(-t/t_r)}{f_n(0) B(0,t) + \int_0^t \exp(-\xi/t_r) B(\xi, t)\, \mathrm{d}\xi}, \tag{40}$$

where

$$f_n(0) = [Q_n(0) + \bar{G}(0)]^{-1} \quad \text{and} \quad B(a,b) = \exp\left(-\int_a^b \bar{G}(\eta)\, \mathrm{d}\eta\right).$$

The localization criterion for the considered case of an adiabatic deformation becomes

$$\frac{\dot{\bar{F}}_2(t)}{\bar{F}_2(t)} - \bar{G}(t) + \frac{\exp(-t/t_r)}{f_n(0) B(0,t) + \int_0^t \exp(-\xi/t_r) B(\xi, t)\, \mathrm{d}\xi} > 0. \tag{41}$$

It is interesting to observe that at the initial time, $t = 0$, the localization criterion associated with the classical Fourier's law requires

$$\frac{\dot{\bar{F}}_2(0)}{\bar{F}_2(0)} + r_1 \sigma \bar{F}_2(0) > 0, \tag{42}$$

while the localization criterion associated with the modified Fourier law requires

$$\frac{\dot{\bar{F}}_2(0)}{\bar{F}_2(0)} + Q_n(0) > 0. \tag{43}$$

The criterion (43) involves the initial values of the temperature perturbations as well as its time derivative. On the other hand, the criterion (42) depends only on the initial homogeneous solution regardless of the initial conditions on the temperature perturbation. In other words, the criterion (42) correlates the onset of localization to the structure of the material's constitutive description without regard to the amplitude of the initial perturbation. However, accounting for the finite speed of thermal wave propagation yields a localization initiation prediction which allows for localization inhibition through appropriate selection of the initial values of the temperature perturbation.

5.3. *Singular perturbation analysis*

The exact solution (40) involves quadratures and applies only to the case of an adiabatic deformation. Further understanding of the role of the relaxation time t_r requires an asymptotic solution which distinguishes the two time scales which are evident from an examination of the solution structure given in eqn (40). It is important to note that the relaxation time t_r is a very small parameter compared to unity. As long as the function $\bar{G}(t)$ is of order one, then eqn (35) exhibits non-oscillatory solutions. Hence, the exact solution is expected to vary rapidly near the initial time $t = 0$ while varying smoothly for large times. Next, we utilize the method of matched asymptotic expansions towards solving the eqn (35) with a small relaxation time. The equation may be rewritten as follows:

$$t_r\ddot{\Theta}_n(t)+[1+t_r\bar{G}(t)]\dot{\Theta}_n(t)+[P(t)+t_r\dot{P}(t)]\Theta_n(t) = 0, \tag{44}$$

where $P(t) \equiv \alpha_n+\bar{G}(t)$.

5.3.1. *Outer solution.* We seek an outer solution which approximates the solution of eqn (44) for times away from the initial time $t = 0$. The outer solution has the series form

$$\Theta_n^0(t\,;t_r) = T_0(t)+t_rT_1(t)+t_r^2T_2(t)+\cdots. \tag{45}$$

Substitution of the expansion (45) into eqn (35) yields the following equations for the zeroth and first order outer solutions:

$$\dot{T}_0(t)+P(t)T_0(t) = 0, \tag{46}$$

$$\dot{T}_1(t)+P(t)T_1(t) = -\alpha_nP(t)T_0(t), \tag{47}$$

The above equations have the following exact solutions:

$$T_0(t) = T_0(0)\mu(0,t), \tag{48}$$

$$T_1(t) = \mu(0,t)\left[T_1(0)-\alpha_nT_0(0)\int_0^t P(\xi)\,\mathrm{d}\xi\right], \tag{49}$$

where $\mu(0,t) = \exp\left[-\int_0^t P(\xi)\mathrm{d}\xi\right]$. It is useful to note that in the absence of heat conduction, the function $\mu(0,t)$ reduces to $B(0,t)$. The integration constants $T_0(0)$ and $T_1(0)$ are to be determined through the matching with the inner solution. The outer solution, accurate to second order, is given by

$$\Theta_n^0(t\,;t_r) = \mu(0,t)\left[T_0(0)+t_r\left\{T_1(0)-\alpha_nT_0(0)\int_0^t P(\xi)\,\mathrm{d}\xi\right\}\right]+O(t_r)^2. \tag{50}$$

5.3.2. *Inner solution.* The inner solution, near $t = 0$, is expected to vary rapidly and therefore it suggests the introduction of a fast variable. A natural choice is

$$u = \frac{t}{t_r}. \tag{51}$$

Making the change of variables (51) reduces eqn (31) to

$$\frac{d^2\Theta_n^i}{du^2}+(1+t_r\bar{G}(t_r u))\frac{d\Theta_n^i}{du}+t_r[P(t_r u)+t_r P'(t_r u)]\Theta_n^i=0, \tag{52}$$

with the associated initial conditions

$$\Theta_n^i(0)=\Theta_0, \frac{d\Theta_n^i(0)}{du}=t_r\dot{\Theta}_0. \tag{53}$$

Note that the first initial conditions is of order one while the second is of order t_r. Now, we seek an inner solution of the power series form

$$\Theta_n^i(u;t_r)=\mathscr{F}_0(u)+t_r\mathscr{F}_1(u)+\cdots. \tag{54}$$

Substitution of the foregoing expansion into eqn (52) and collecting the coefficients of equal powers of t_r gives†

$$\left.\begin{aligned}&\frac{d^2\mathscr{F}_0}{du^2}+\frac{d\mathscr{F}_0}{du}=0,\\ &\mathscr{F}_0(0)=\Theta_0,\quad \frac{d\mathscr{F}_0(0)}{du}=0,\end{aligned}\right\}\text{“Zeroth-order system”}, \tag{55}$$

and

$$\left.\begin{aligned}&\frac{d^2\mathscr{F}_1}{du^2}+\frac{d\mathscr{F}_1}{du}=-\bar{G}(0)\frac{d\mathscr{F}_0}{du}-P(0)\mathscr{F}_0,\\ &\mathscr{F}_1(0)=0,\quad \frac{d\mathscr{F}_1(0)}{du}=\dot{\Theta}_0\end{aligned}\right\}\text{“First-order system”}. \tag{56}$$

The exact solutions for the functions $\mathscr{F}_0(u)$ and $\mathscr{F}_1(u)$ are given by

$$\mathscr{F}_0(u)=\Theta_0, \tag{57}$$

$$\mathscr{F}_1(u)=\Theta_0[\{P(0)+Q_n(0)\}(1-e^{-u})-uP(0)]. \tag{58}$$

The inner solution, accurate to second order, is given by

$$\Theta_n^i(u;t_r)=\Theta_0\{1+t_r[\{P(0)+Q_n(0)\}(1-e^{-u})-uP(0)]\}. \tag{59}$$

5.3.3. *Matching*. We utilize the van Dyke's matching technique to match the inner and outer solutions which leads to the determination of the two unknown constants $T_0(0)$ and $T_1(0)$. For this purpose, the two-term inner expansion of the outer solution is matched with

† Note that the time-dependent coefficient functions in eqn (52) can be expanded as follows:

$$\bar{G}(t_r u)=\bar{G}(0)+t_r\bar{G}'(0)u+O(t_r^2),$$

$$P(t_r u)=P(0)+t_r P'(0)u+O(t_r^2),$$

$$P'(t_r u)=P'(0)+t_r P''(0)u+O(t_r^2),$$

where a “prime” denotes differentiation with respect to the enclosed function argument.

the two-term outer expansion of the inner solution. The two-term inner expansion of the outer solution is given by†

$$(\Theta_n^0)^i(ut_r\,;t_r) = T_0(0)+t_r\{T_1(0)-T_0(0)P(0)u\}+O(t_r)^2 \tag{60}$$

while the two-term outer expansion of the inner solution is given by

$$(\Theta_n^i)^0(t/t_r\,;t_r) = \Theta_0\left[1+t_r\{[P(0)+Q_n(0)](1-\mathrm{e}^{-t/t_r})\}-\frac{t}{t_r}P(0)\right]+O(t_r^2). \tag{61}$$

The van Dyke matching principle requires that

$$\lim_{t_r\to 0}[(\Theta_n^0)^i(t\,;t_r)] = \lim_{t_r\to 0}[(\Theta_n^i)^0(t/t_r\,;t_r)] \tag{62}$$

which provides the two unknown constants

$$T_0(0) = \Theta_0,\ T_1(0) = \Theta_0[P(0)+Q_n(0)]. \tag{63}$$

The composite solution, that is uniformly valid for all t, is given by

$$\Theta_n^c(t) = \Theta_n^0(t)+\Theta_n^i(t)-(\Theta_n^i)^0(t)$$

which upon substitution of eqns (50), (59) and (61) reduces to

$$\Theta_n^c(t) = \Theta_0\mu(0,t)\left[1+t_r\left\{[P(0)+Q_n(0)](1-\mu^{-1}(0,t)\mathrm{e}^{-t/t_r})-\alpha_n\int_0^t P(\xi)\,\mathrm{d}\xi\right\}\right]. \tag{64}$$

It is evident that the contribution of the inner solution to the uniformly valid composite solution (64) is of order t_r. However, the contribution of the inner solution to the time derivative of the composite solution is significant at early times. The time derivative of the composite solution is given by

$$\frac{\dot{\Theta}_n^c(t)}{\Theta_n^c(t)} = -P(t)+[P(0)+Q_n(0)]\frac{\mathrm{e}^{-t/t_r}}{\mu(0,t)}+O(t_r). \tag{65}$$

Upon substitution of the foregoing quotient into the localization criterion (34), we obtain

$$\frac{\dot{\bar{F}}_2(t)}{\bar{F}_2(t)}-P(t)+[P(0)+Q_n(0)]\frac{\mathrm{e}^{-t/t_r}}{\mu(0,t)} > 0. \tag{66}$$

At the initial time $t = 0$, the localization criterion (66) reduces to

† Note that

$$\mu(0,t) = \mu(0,t_r u) = 1-t_rP(0)u+O(t_r^2),$$

$$\int_0^t P(\xi)\,\mathrm{d}\xi = t_rP(0)u+O(t_r^2).$$

$$\frac{\dot{\bar{F}}_2(0)}{\bar{F}_2(0)}+Q_n(0) > 0$$

which is identical to the condition (43) derived earlier for the case of an adiabatic deformation.

5.4. *Application to power law hardening materials*

Consider a strain-independent material whose thermal viscoplastic response is modeled by the empirical power law

$$\sigma = \theta^{v}(\dot{\gamma}^{p})^{m}, \tag{67}$$

where $v < 0$ is the thermal softening exponent while $m > 0$ is the strain-rate hardening exponent. Upon substitution of the homogeneous temperature solution

$$\bar{\theta}(t) = [1+r_1(1-v)t]^{1/1-v}$$

into expression (66), we obtain the relevant localization criterion

$$h(t;t_r,\alpha_n) \equiv r_1(p-1)[1-L(t;t_r,\alpha_n)]-\alpha_n f(t)+[\alpha_n+Q_n(0)-r_1]L(t;t_r,\alpha_n) > 0, \tag{68}$$

where

$$L(t;t_r,\alpha_n) \equiv \mathrm{e}^{[\alpha_n-(1/t_r)]t}f(t)^{1-p-v/1-v},\quad f(t) \equiv 1+r_1(1-v)t,\quad p \equiv -\frac{v}{m}. \tag{69}$$

We note that $L(0;t_r,\alpha_n) = 1$ while $L(t;0,\alpha_n) = 0$. Hence, we may isolate the following limiting localization criteria:

Fourier law	Diffusion	Localization criterion		
		$t = 0$	$t > 0$	$t/t_r \to \infty$
Classical	$r_0 \neq 0$	$Z-\alpha_n > 0$	$Z-\alpha_n f(t) > 0$	N/A
	$r_0 = 0$	$p > 1$	$p > 1$	N/A
Modified	$r_0 \neq 0$	$Q_n(0) > r_1$	$h(t;t_r,\alpha_n) > 0$	$Z-\alpha_n f(t) > 0$
	$r_0 = 0$	$Q_n(0) > r_1$	$h(t;t_r,0) > 0$	$p > 1$

where $Z \equiv r_1(p-1)$.

6. ANALYSIS AND CONCLUSIONS

We examine the implications of the various localization criteria compiled in the table at the end of Section 5.4. Hence, we make the following general observations:

1. General observations

(a) The localization initiation criterion associated with the classical Fourier law is obtained from (68) by setting $t_r = 0$. This gives

$$h_c(t;\alpha_n) \equiv r_1(p-1)-\alpha_n f(t) > 0. \tag{70}$$

(b) The necessary condition for localization initiation is independent of the initial conditions for the classical Fourier law while it depends on the quotient $Q_n(0) = \dot{\Theta}_n(0)/\Theta_n(0)$ for the modified Fourier law.

(c) The necessary localization condition associated with the modified Fourier law reduces to that associated with the classical Fourier law for large values of t/t_r. Hence, the effect of the modified Fourier law is restricted to the early times ($t/t_r \sim O(1)$).

2. Adiabatic deformation and small relaxation time

(a) For sufficiently small values of the relaxation time t_r combined with an adiabatic deformation, the left side of the localization inequality (68) reduces to

$$\lim_{t_r \ll 1, \alpha_n = 0} h(t; t_r, \alpha_n) \equiv H(u) = r_1(p-1) + [Q_n(0) - r_1 p]e^{-u}. \tag{71}$$

The behavior of the above function is illustrated in Fig. 2. The profiles illustrated in Fig. 2 are obtained for values of the quotient $Q_n(0)$ in the range $[-10, 18]$ with an increment of "4". These profiles are computed for a power-law material with the following dimensionless numerical values for various parameters: $v = -0.38$, $m = 0.019$, $r_1 = 0.5$ and, $r_0 = 0$. It is important to observe that the function $H(u)$ does not depend explicitly on the relaxation time t_r. For small values of t_r, the time interval over which those profiles change before they approach their asymptotic limit, $r_1(p-1)$, is rather small. In fact, this behavior suggests that the effects of modifying the Fourier law are constrained to a very narrow initial interval for which $u = t/t_r < 5$.

(b) For $p > 1$ (that is; $v+m < 0$), the asymptotic long-time limit of the function $H(u)$ is $r_1(p-1) > 0$. As indicated in the Fig. 2, the $H(u)$ profiles corresponding to $Q_n(0) < r_1$ are initially negative with a positive slope. Each of these profiles exhibit a critical value of u, called u_{cr}, after which it attains positive values. The critical u is given by

$$u_{cr} = \log\left[\frac{r_1 p - Q_n(0)}{r_1(p-1)}\right], \quad p > 1, \quad Q_n(0) < r_1. \tag{72}$$

The critical time u_{cr} signals the onset of shear localization following an initial retardation of localization by the thermal waves effects.

3. Diffusion effects

(a) The necessary localization condition for the classical Fourier law with heat conduction effects predicts an initial wavelength threshold below which perturbations do not localize. Such an initial wavelength threshold is absent from the localization condition associated with the modified Fourier law. This observation is strictly applicable to the initial time $t = 0$.

(b) The critical value of the local adiabaticity parameter α_n, called α_{cr}, is given by

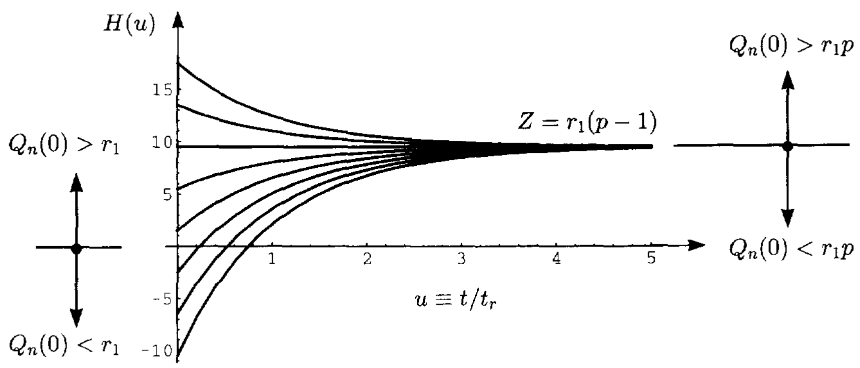

Fig. 2. Behavior of the localization function $H(u)$.

$$\alpha_{cr} = r_0 \xi_{cr}^2 = r_1 (p-1). \tag{73}$$

The foregoing expression combined with the criterion (70) indicate that all wave numbers (lengths) greater (smaller) than ξ_{cr} ($l_{cr} = 1/\xi_{cr}$) are stable.

(c) The behavior of the function $h(t\,; t_r, \alpha_n)$ for values of $\alpha_n < \alpha_{cr}$ is illustrated in Fig. 3. This figure illustrates the dependence of the localization initiation criterion on the quotient $Q_n(0)$ as well as the local adiabaticity parameter α_n. The shown profiles are computed using the same numerical values as in Fig. 2 with the addition of $t_r = 0.01$. The value of α_{cr} for the considered computation is "9.5". Furthermore, the shaded plane in each figure corresponds to $h = 0$. Therefore, portions of the h-surface below the shaded plane correspond to a stable response while those above the plane correspond to a localizing response. Fig. 4 illustrates the behavior of the function $h(t\,; t_r, \alpha_n)$ for values of $\alpha_n > \alpha_{cr}$.

(d) It is important to observe that, for the classical Fourier law, each of the three-dimensional graphs will reduce to a plane surface whose height is "$Z - \alpha_n f(t)$" independent of the axis $Q_n(0)$. Hence, in contrast to the classical Fourier law, the present model predicts some initial retardation of localization for some range of the initial values and for $\alpha_n < \alpha_{cr}$. On the other hand, the model also predicts the possibility of initial localization for some range of the initial values and for $\alpha_n > \alpha_{cr}$.

(e) Figure 5 shows the contours of $h(t\,; t_r, \alpha_n) = 0$ in a $t - Q_n(0)$ plane for various values of $\alpha_n \in [0, 20]$. Examination of Fig. 5 indicates that

- *For* $\alpha_n \leqslant 9.1$: The delay in localization initiation, for a fixed negative value of $Q_n(0)$, increases with increasing wave numbers.
- *For* $\alpha_n \leqslant 9.4$: The initial localization, for a fixed positive value of $Q_n(0)$, is suppressed faster for larger wave numbers.

In closing, it is useful to note that typical relaxation times for structural steels are of the order of 10^{-9} seconds. Dimensionless values of the relaxation time t_r are of the order of 10^{-6} for Kolsky bar tests while they are of the order of 10^{-4}–10^{-2} for plate impact experiments. Hence, it is reasonable to conclude that the effect of modifying the Fourier

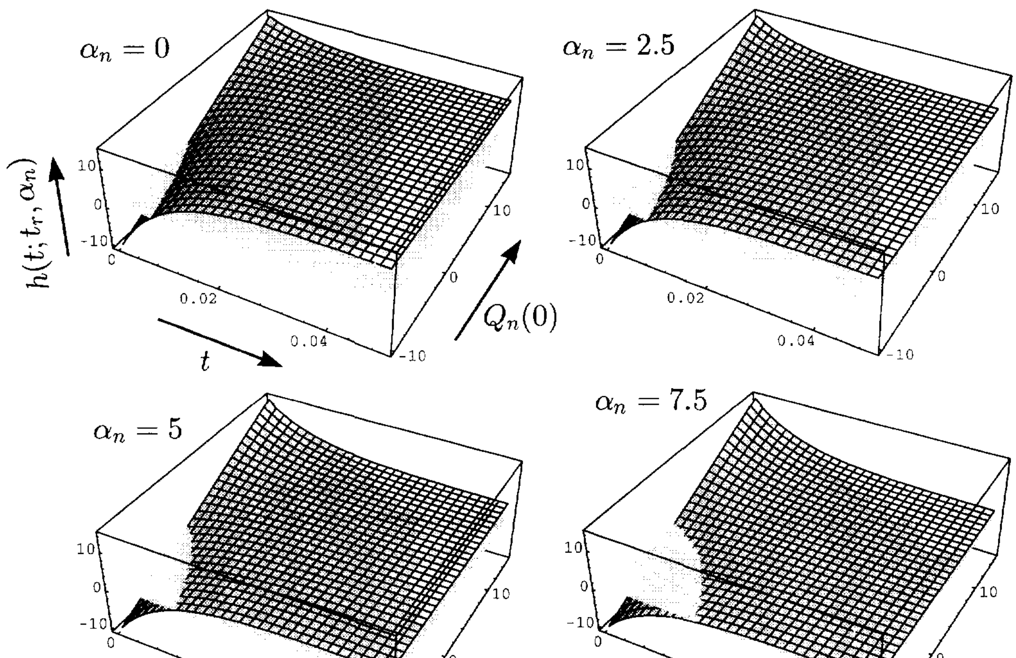

Fig. 3. Behavior of the function $h(t\,; t_r, \alpha_n)$ for $\alpha_n < \alpha_{cr}$.

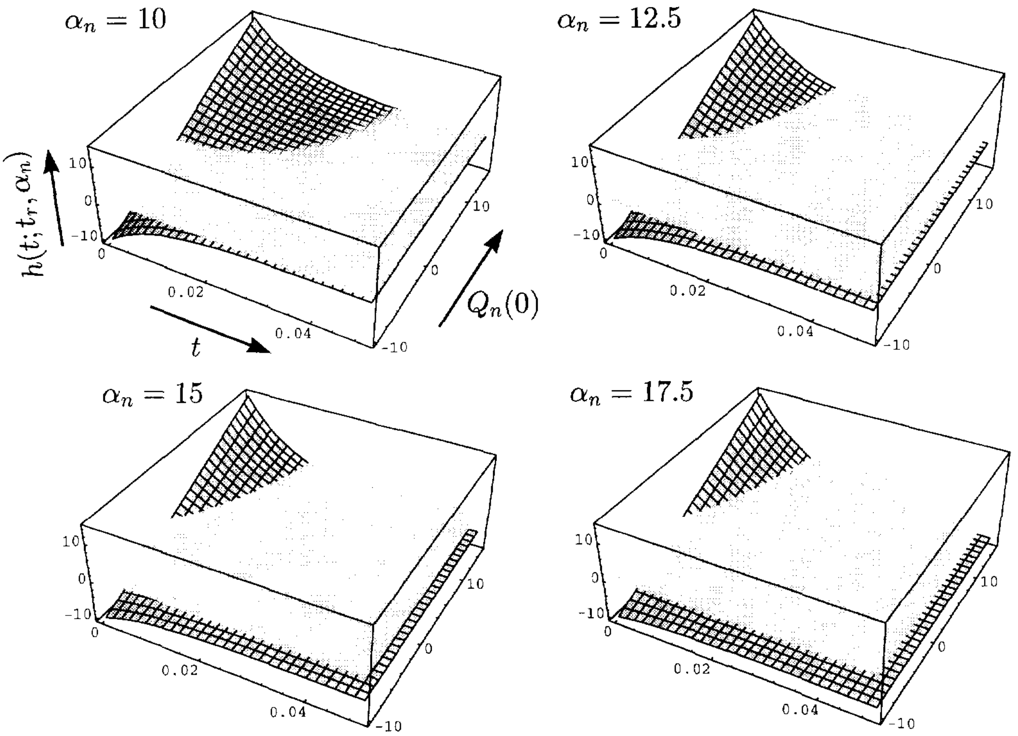

Fig. 4. Behavior of the function $h(t; t_r, \alpha_n)$ for $\alpha_n > \alpha_{cr}$.

law on localization initiation is very weak. In other words, the effects of the finite speed of thermal wave propagation on the onset of shear flow localization can be ignored for deformations at the applied strain rates of 10^5 sec^{-1} or less. On the other hand, for plate impact tests, the observation times are often of the order of the time during which the elastic waves can traverse the specimen two or three times. Since the thermal wave speed is much smaller than the longitudinal wave speed, this implies that the thermal waves may be of importance in plate impact problems. For ballistic applications, the dimensionless

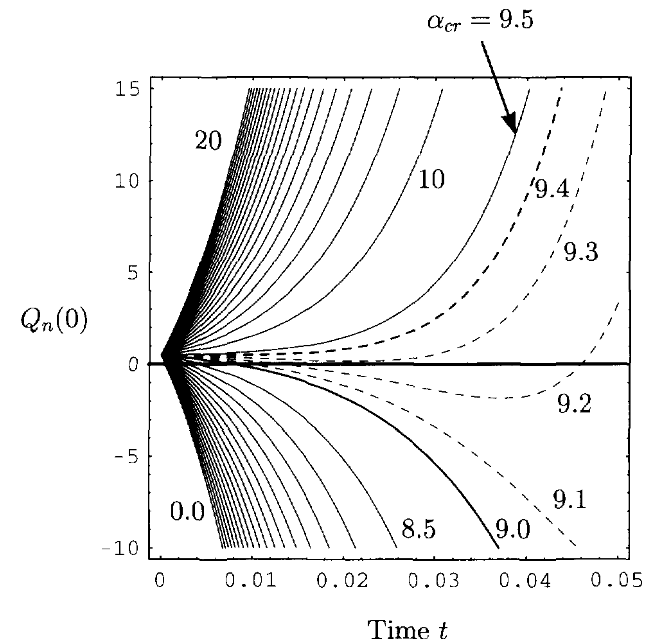

Fig. 5. Contours of $h(t; t_r, \alpha_n) = 0$ for various values of α_n.

relaxation time is of order one and, therefore, the results of the asymptotic analysis of the present paper cannot be used.

The finite wave speed of thermal wave propagation is expected to have a significant effect on localization evolution during severe localization in view of the large local strain rates within the localized zones. Further work is required to resolve this issue.

Acknowledgements—The second author (TGS) would like to appreciate the support of the National Science Foundation through the Presidential Young Investigator Award #NSF MSS 89-57180 PYI. The first author (HPC) would like to thank Prof. Christopher Lawrence for many helpful discussions.

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